Algebra of Sound Waves – Problem Set

Frequency Ratios, Harmonic Intervals, and Beat Frequencies for Physics Exams

20 questions • Increasing difficulty • Real-world audio scenarios

For use with: A-Level, AP Physics, IB, and introductory acoustics courses.

Section 1: Basic Frequency Ratios (Questions 1–5)

Q1 – Octave ratio

The frequency of A4 is 440 Hz. What is the frequency of A5 (one octave higher)?

Q2 – Perfect fifth

A perfect fifth has a frequency ratio of 3:2 (higher : lower).

If the lower note is 220 Hz, what is the frequency of the perfect fifth above it?

Q3 – Perfect fourth

A perfect fourth has a ratio of 4:3.

If the higher note is 480 Hz, what is the frequency of the lower note?

Q4 – Major third

A major third has a ratio of 5:4.

Two notes form a major third. The lower note is 330 Hz. What is the frequency of the higher note?

Express the answer to one decimal place.

Q5 – Real-world scenario: Tuning a piano

A piano string should produce 261.63 Hz (middle C). The actual frequency is 264 Hz.

What is the frequency ratio (actual / desired)?

Is the string sharp or flat? By approximately how many cents?

(Use cents = 1200 × $\log_2$(f_actual/f_desired).)

Section 2: Complex Intervals and Ratio Inversions (Questions 6–10)

Q6 – Minor sixth inversion

A minor sixth is the inversion of a major third. If a major third has ratio 5:4, what is the ratio of a minor sixth (lower : higher)?

(Hint: Inversion = octave divided by original ratio, so ratio = 2 / (5/4) = 8/5.)

Q7 – From ratio to interval name

A frequency ratio of 9:8 is a major second.

If two notes have frequencies 400 Hz and 450 Hz, what interval do they form?

Confirm using the ratio.

Q8 – Real-world scenario: String length and frequency

For a vibrating string, frequency is inversely proportional to length ($f \propto 1/L$).

A string of length 1.0 m produces 110 Hz. What length produces a perfect fifth above that note?

(Hint: perfect fifth ratio = 3/2; new frequency = 110 × 3/2 = 165 Hz; then L = original length × (original freq / new freq) = 1.0 × (110/165) = 0.6667 m.)

Q9 – Just intonation vs. equal temperament

In just intonation, a major third is 5:4 = 1.25. In equal temperament, a major third is $2^{(4/12)} \approx 1.25992$.

Calculate the difference in cents between the two.

(Use cents = $1200 \times \log_2(\text{ratio\_EQ} / \text{ratio\_just})$.)

Q10 – Harmonic series

A fundamental frequency of 100 Hz has harmonics at integer multiples (100, 200, 300, 400, 500, … Hz).

Which harmonics form a perfect fifth above the fundamental? A perfect fourth? Express as ratios relative to the fundamental.

Section 3: Beat Frequencies and Interference (Questions 11–15)

Q11 – Basic beat frequency

Two tuning forks have frequencies 256 Hz and 260 Hz.
What is the beat frequency?

Q12 – Real-world scenario: Piano tuning using beats

A piano tuner strikes a tuning fork of 440 Hz and a piano key simultaneously. She hears 2 beats per second.
What are the two possible frequencies of the piano string?

Q13 – Beat frequency from mistuned unison

Two flutes both try to play A4 = 440 Hz. One is 441 Hz, the other is 439 Hz.
What beat frequency results?
If a third flute at 440.5 Hz joins, what is the beat frequency between the first and third?

Q14 – Phase and interference type

Two speakers emit 500 Hz waves. At a listener's position, the path difference is 0.34 m. Speed of sound = 340 m/s.
Calculate the phase difference in radians. Is the interference constructive, destructive, or intermediate?

Q15 – Real-world scenario: Acoustic guitar feedback

A guitar string vibrates at 246 Hz. A nearby loudspeaker emits a 246.5 Hz tone.
What beat frequency will be heard?
If the beat frequency is < 3 Hz, is it likely to cause audible flutter? Why?

Section 4: Advanced – Multiple Intervals, Sum and Difference Tones (Questions 16–20)

Q16 – Sum and difference tones (nonlinear distortion)

When two tones of frequencies f_1 and f_2 are played loudly, the human ear can perceive distortion products at $f_1 + f_2$ and $|f_1 - f_2|$.
For $f_1 = 440$ Hz, $f_2 = 660$ Hz (perfect fifth), calculate the two distortion frequencies.
Which one is within the audible range (20 – 20,000 Hz)?

Q17 – Real-world scenario: Organ pipe length ratios

An open pipe's fundamental frequency is $f = v/(2L)$. A closed pipe's fundamental is $f = v/(4L)$.
An open pipe produces 262 Hz. What length of closed pipe gives the same frequency? ($v = 343$ m/s)
Then, using length ratios, what is the ratio of open to closed pipe lengths for a perfect fifth interval (3:2 frequency ratio)?

Q18 – Interval stacking

A perfect fifth (3:2) stacked on top of another perfect fifth gives a ratio of $(3/2)^2$ = 9/4.

What interval does 9/4 represent when reduced to within one octave?

(Hint: divide by 2 to bring within [1,2] → 9/8, which is a major second.)

Q19 – Beat frequency from two intervals

A piano plays two notes: 440 Hz (A4) and 554.37 Hz (C#5, a major third above). The third harmonic of the lower note (3×440 = 1320 Hz) and the second harmonic of the higher note (2×554.37 = 1108.74 Hz) are close but not equal.

Calculate the beat frequency between these two harmonics.

Is this likely to cause audible roughness?

Q20 – Master problem: Interval ratio and cents calculation

A musician plays 440 Hz and 445 Hz.

a) What is the frequency ratio (higher/lower)?

b) How many cents apart are these two frequencies?

c) Name the closest equal-temperament interval (e.g., minor second, major second, etc.) given that a semitone = 100 cents.

d) If the musician wants the interval to be a perfect 25:24 ratio (a just chromatic semitone ≈ 70.7 cents), how many cents sharp or flat is the 445 Hz note from the just target?

Answer Key

Section 1

Q1

A5 = 440 × 2 = **880 Hz**.

Q2

Higher = lower × (3/2) = 220 × 1.5 = **330 Hz**.

Q3

Higher / lower = 4/3 → lower = higher × (3/4) = 480 × 0.75 = **360 Hz**.

Q4

Higher = 330 × (5/4) = 330 × 1.25 = **412.5 Hz**.

Q5

Ratio = 264 / 261.63 ≈ **1.00906**. Sharp (above desired).

Cents = 1200 × $\log_2$(1.00906) ≈ 1200 × (ln 1.00906 / ln 2) ≈ 1200 × (0.00902 / 0.693147) ≈ 1200 × 0.01301 ≈ **15.6 cents**. (A semitone = 100 cents, so about 1/6 of a semitone sharp.)

Section 2

Q6

Inversion ratio = 2 / (5/4) = 2 × 4/5 = **8/5** (lower : higher = 8:5). That's a minor sixth.

Q7

Ratio = 450 / 400 = **9/8** = 1.125. That is a **major second**.

Q8

Desired frequency = 110 × 3/2 = 165 Hz.

Since f ∝ 1/L, L_new = L_old × (f_old / f_new) = 1.0 × (110/165) = **0.6667 m** (2/3 m).

Q9

Ratio_EQ / ratio_just = 1.25992 / 1.25 = 1.007936.

Cents = 1200 × $\log_2$(1.007936) ≈ 1200 × (0.007905 / 0.693147) ≈ 1200 × 0.01141 ≈ **13.7 cents**. (Equal-temperament major third is about 14 cents wider than just.)

Q10

Perfect fifth above fundamental = 3:2 ratio → 3rd harmonic (300 Hz) is a perfect fifth above 200 Hz? Wait, careful: Fundamental 100 Hz. Fifth above is 150 Hz. The 150 Hz is not a harmonic of 100 Hz (harmonics are 100,200,300,400...). But the 3rd harmonic (300 Hz) is a perfect twelfth (octave + fifth) above 100 Hz. So: perfect fifth above fundamental is not an integer harmonic. However, the ratio 3:2 appears between harmonics: 300 Hz (3rd) and 200 Hz (2nd) gives 3:2. So the interval between the 2nd and 3rd harmonics is a perfect fifth. Similarly, perfect fourth (4:3) between 3rd and 4th harmonics (400 Hz / 300 Hz = 4:3). So: **perfect fifth = 3rd harmonic relative to 2nd harmonic; perfect fourth = 4th harmonic relative to 3rd harmonic.**

Section 3

Q11

Beat frequency = |260 – 256| = **4 Hz**.

Q12

|piano – 440| = 2 → piano = **442 Hz** or **438 Hz**.

Q13

Between 441 and 439: beat = |441 – 439| = **2 Hz**.

Between 441 and 440.5: beat = **0.5 Hz**.

Q14

$\lambda = v/f = 340/500 = 0.68$ m. $\Delta x = 0.34$ m $= \lambda/2$.

$\Delta\varphi = 2\pi (\Delta x/\lambda) = 2\pi \times 0.5 =$ **π radians (180°)**.

Destructive interference.

Q15

Beat = |246.5 – 246| = **0.5 Hz**.

Yes, 0.5 Hz is < 3 Hz and will cause a slow "wow" flutter, very audible and distracting.

Section 4

Q16

Sum = 440 + 660 = **1100 Hz**. Difference = |440 – 660| = **220 Hz**. Both are within 20–20,000 Hz (220 Hz and 1100 Hz are audible). The difference tone (220 Hz) is often perceived as a low bass note.

Q17

Open pipe: $f = v/(2L) \rightarrow L_{open} = v/(2f) = 343/(2\times262) \approx 343/524 \approx 0.6546$ m.

Closed pipe: $f = v/(4L) \rightarrow L_{closed} = v/(4f) = 343/(4\times262) \approx 343/1048 \approx 0.3273$ m.

For perfect fifth (3:2 frequency ratio) using same type of pipe (both open or both closed), length ratio = inverse of frequency ratio = 2/3 (shorter pipe gives higher pitch).

For open: $L_{long} / L_{short} = 3/2$? Wait, careful: $f_{long} / f_{short} = 2/3$ if long is lower? Simpler: for a given pipe type, $L \propto 1/f$. So if $f2/f1 = 3/2$, then $L1/L2 = 3/2$. So the longer pipe is 1.5× the shorter pipe.

Q18

Stack two perfect fifths: $(3/2)^2 = 9/4 = 2.25$. Reduce to within 1–2 by dividing by 2 → 9/8 = 1.125, which is a **major second**.

Q19

Harmonic frequencies: 3×440 = 1320 Hz; 2×554.37 = 1108.74 Hz.

Beat = |1320 – 1108.74| = **211.26 Hz**.

This is not a slow beat; it's a high difference tone, likely not perceived as roughness but as a new pitch (around 211 Hz, close to G#3). Not typically problematic.

Q20

a) Ratio = 445/440 = **1.0113636**.

b) Cents = $1200 \times \log_2(1.0113636) = 1200 \times (\ln 1.0113636 / \ln 2) \approx 1200 \times (0.01130 / 0.693147) \approx 1200 \times 0.01630 \approx$ **19.56 cents**.

c) Closest equal-temperament interval: a semitone = 100 cents, a quarter tone = 50 cents. 19.6 cents is closest to a **chromatic semitone** (100 cents? No, that's too big). Actually, 19.6 cents is less than half a quarter tone; it's roughly a **comma**. No standard Western interval name except "very narrow major second"? But given options: It's slightly less than a quarter tone (50 cents). Could be called a **"small diesis"**. For exam purposes: **"about 1/5 of a semitone"** – it's a microtonal interval.

d) Just chromatic semitone ratio 25/24 ≈ 1.041667. From 440 Hz, just target = 440 × 25/24 ≈ 458.33 Hz. Our note is 445 Hz, which is **13.33 Hz lower**. In cents: $1200 \times \log_2(445/458.33) = 1200 \times \log_2(0.9709) \approx -51.2$ cents. So 445 Hz is **51 cents flat** of the just chromatic semitone above 440 Hz.

End of Problem Set

Created by Decroly Education Centre for use with "Perfect Fifths Aren't Magic | They're Pure Mathematics" *explainer video.*

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